Characterization of queer super crystals

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based on joint work with Maria Gillespie, Graham Hawkes, Wencin Poh

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Goal

- **Lie superalgebras**: arose in physics to unify bosons and fermions
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- In mathematics: projective representations of the symmetric group
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- Highest weight crystals for queer super Lie algebras (Grantcharov et al.)
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- **Lie superalgebras**: arose in physics to unify bosons and fermions
- In mathematics: projective representations of the symmetric group
- Queer super Lie algebra
- Highest weight crystals for queer super Lie algebras (Grantcharov et al.)
- Characterization of these crystals and how these discoveries were guided by experimentation with SageMath
Outline

1. Crystals of type $A_n$
2. Queer supercrystals
3. Stembridge axioms
4. Characterization of queer crystals
Crystals of type $A_n$

Abstract crystal of type $A_n$: nonempty set $B$ together with the maps

\[ e_i, f_i : B \to B \sqcup \{0\} \quad (i \in I) \]
\[ \text{wt} : B \to \Lambda \]
Crystals of type $A_n$

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$$e_i, f_i : B \to B \sqcup \{0\} \quad (i \in I)$$

$$\text{wt} : B \to \Lambda$$

weight lattice $\Lambda = \mathbb{Z}^{n+1}_{\geq 0}$
index set $I = \{1, 2, \ldots, n\}$
simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\epsilon_i$ $i$-th standard basis vector of $\mathbb{Z}^{n+1}$

We require:

A1. $f_i b = b'$ if and only if $b = e_i b'$

$$\text{wt}(b') = \text{wt}(b) + \alpha_i$$
Crystals of type $A_n$

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simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $\epsilon_i$ $i$-th standard basis vector of $\mathbb{Z}^{n+1}$

string lengths for $b \in B$

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(b) \neq 0\}$$

$$\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(b) \neq 0\}$$
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We require:

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Crystal: $A_n$ example

Example

Standard crystal $B$ for type $A_n$:
Crystal: $A_n$ example

Example

Standard crystal $B$ for type $A_n$:

- Weight function $\text{wt} ([i]) = \epsilon_i$
- Highest weight element: 1
Tensor products

$B$ and $C$ crystals of type $A_n$

**Definition**

Tensor product $B \otimes C$ has the following data:
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- **Elements:** $b \otimes c := (b, c) \in B \times C$
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**Tensor product** $B \otimes C$ has the following data:

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Tensor product $B \otimes C$ has the following data:

- **Elements**: $b \otimes c := (b, c) \in B \times C$
- **Weight map**: $\text{wt}(b \otimes c) = \text{wt}(b) + \text{wt}(c)$
- **Crystal operators**:

\[
\begin{align*}
  f_i(b \otimes c) &= \begin{cases} 
  f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c) \\
  b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) 
  \end{cases} \\
  e_i(b \otimes c) &= \begin{cases} 
  e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \\
  b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c) 
  \end{cases}
\end{align*}
\]
Example: Tensor product

Components of crystal of words $B^\otimes 3 = B \otimes B \otimes B$ of type $A_2$:
Motivation

Why are crystals interesting?
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Why are crystals interesting?

- **Characters**: character of highest weight crystal \( B(\lambda) \) is the Schur function \( s_\lambda \)
Motivation

Why are crystals interesting?

- **Characters**: character of highest weight crystal $B(\lambda)$ is Schur function $s_{\lambda}$
- **Littlewood–Richardson rule**: 
  
  $$s_{\lambda}s_{\mu} = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}$$

  $c_{\lambda\mu}^{\nu} = \text{number of highest weights of weight } \nu \text{ in } B(\lambda) \otimes B(\mu)$
Outline

1. Crystals of type $A_n$
2. Queer supercrystals
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4. Characterization of queer crystals
Queer crystal: Developments

- Queer Lie superalgebra $q(n)$: a super analogue of $\mathfrak{gl}(n)$
Queer crystal: Developments

- **Queer Lie superalgebra** $q(n)$: a super analogue of $\mathfrak{gl}(n)$
- [Grantcharov, Jung, Kang, Kashiwara, Kim, ’10]: Crystal basis theory for queer Lie superalgebras using $U_q(q(n))$
  - Introduced queer crystals on words with tensor product rule.
  - Explicit combinatorial realization of queer crystals using semistandard decomposition tableaux.
  - Existence of fake highest (and lowest) weights on queer crystals.
Standard crystal and tensor product

Example

Standard queer crystal $\mathcal{B}$ for $q(n + 1)$

\[
\begin{align*}
1 & \quad \rightarrow_{1} \quad 2 \quad \rightarrow_{2} \quad 3 \quad \rightarrow_{3} \cdots \rightarrow_{n} \quad n + 1 \\
& \quad \leftarrow_{-1} \quad \rightarrow_{1} \quad \rightarrow_{2} \quad \rightarrow_{3} \quad \cdots \quad \rightarrow_{n} \\
\end{align*}
\]
Standard crystal and tensor product

Example

Standard queer crystal $\mathcal{B}$ for $q(n + 1)$

\[
\begin{array}{c}
1 \\
-1
\end{array} \quad \rightarrow \quad 2 \quad \rightarrow \quad 3 \quad \rightarrow \quad \cdots \quad \rightarrow \quad n + 1
\]

Tensor product: $b \otimes c \in B \otimes C$

\[
f_{-1}(b \otimes c) = \begin{cases} 
    b \otimes f_{-1}(c) & \text{if } \text{wt}(b)_1 = \text{wt}(b)_2 = 0 \\
    f_{-1}(b) \otimes c & \text{otherwise}
\end{cases}
\]

\[
e_{-1}(b \otimes c) = \begin{cases} 
    b \otimes e_{-1}(c) & \text{if } \text{wt}(b)_1 = \text{wt}(b)_2 = 0 \\
    e_{-1}(b) \otimes c & \text{otherwise}
\end{cases}
\]
Queer crystal: Example

One connected component of $B^\otimes 4$ for $q(3)$:
Motivation

Why are queer crystals interesting?

- **Characters:**
  - character of highest weight crystal $B(\lambda)$ ($\lambda$ strict partition) is *Schur P function* $P_\lambda$
Motivation

Why are queer crystals interesting?

- **Characters:**
  character of highest weight crystal $B(\lambda)$ ($\lambda$ strict partition) is *Schur* $P$ function $P_\lambda$

- **Littlewood–Richardson rule:**

$$P_\lambda P_\mu = \sum_\nu g^\nu_{\lambda\mu} P_\nu$$

$g^\nu_{\lambda\mu} =$ number of highest weights of weight $\nu$ in $B(\lambda) \otimes B(\mu)$
(Fake) highest weight elements

In the queer crystal there exist fake highest weight elements. 
**Question:** How do we detect highest weight elements?
(Fake) highest weight elements

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**Definition**

\[
f_{-i} := s_{w_i}^{-1}f_{-1}s_{w_i} \quad \text{and} \quad e_{-i} := s_{w_i}^{-1}e_{-1}s_{w_i}
\]

where \( w_i = s_2 \cdots s_is_1 \cdots s_{i-1} \) and \( s_i \) is the reflection along the \( i \)-string.
(Fake) highest weight elements

In the queer crystal there exist fake highest weight elements.

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**Definition**

\[ f_{-i} := s_{w_{-1}}f_{-1}s_{w_i} \quad \text{and} \quad e_{-i} := s_{w_{-1}}e_{-1}s_{w_i} \]

where \( w_i = s_2 \cdots s_is_1 \cdots s_{i-1} \) and \( s_i \) is the reflection along the \( i \)-string

**Theorem** (Grantcharov et al. 2014)

*Each connected component in \( B^\otimes \ell \) has a unique highest weight element with*

\[ e_iu = 0 \quad \text{and} \quad e_{-i}u = 0 \quad \text{for all} \ i \in l_0 = \{1, 2, \ldots, n\} \]
(Fake) highest weight elements

In the queer crystal there exist fake highest weight elements.  

**Question:** How do we detect highest weight elements?

**Definition**

\[
 f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i} \quad \text{and} \quad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}
\]

where \( w_i = s_2 \cdots s_i s_1 \cdots s_i^{-1} \) and \( s_i \) is the reflection along the \( i \)-string.

**Theorem (Grantcharov et al. 2014)**

*Each connected component in \( B^\otimes \ell \) has a unique highest weight element with*

\[
e_i u = 0 \quad \text{and} \quad e_{-i} u = 0 \quad \text{for all} \quad i \in I_0 = \{ 1, 2, \ldots, n \}
\]

Similarly

\[
 f_{-i'} := s_{w_0} e_{-(n+1-i)} s_{w_0} \quad \text{and} \quad e_{-i'} := s_{w_0} f_{-(n+1-i)} s_{w_0}
\]

where \( w_0 \) is long word in \( S_{n+1} \), give lowest weight elements.
Queer crystal: Example revisited

Same connected component of $B^\otimes 4$:
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Question

Is there a **local characterization** of a crystal graph?
Stembridge axioms

**Question**

Is there a local characterization of a crystal graph?

- [Stembridge '03] Yes, for crystals of simply-laced root systems (in particular type $A_n$)
- Local rules characterize Stembridge crystals: allows pure combinatorial analysis of these crystals
Stembridge axioms

$B$ crystal for a simply-laced root system with index set $I = \{1, 2 \ldots, n\}$.

**Axiom S1.** For distinct $i, j \in I$ and $x, y \in B$ with $y = e_i x$, then either $\varepsilon_j(y) = \varepsilon_j(x) + 1$ or $\varepsilon_j(y) = \varepsilon_j(x)$.
Stembridge axioms

Axiom S2. For distinct \( i, j \in I \), if \( x \in B \) with both \( \varepsilon_i(x) > 0 \) and \( \varepsilon_j(x) = \varepsilon_j(e_i x) > 0 \), then \( e_i e_j x = e_j e_i x \) and \( \varphi_i(e_j x) = \varphi_i(x) \).
Axiom S3. For distinct $i, j \in I$, if $x \in B$ with both
\[ \varepsilon_j(e_ix) = \varepsilon_j(x) + 1 > 1 \quad \text{and} \quad \varepsilon_i(e_jx) = \varepsilon_i(x) + 1 > 1, \]
then $e_i e_j e_i x = e_j e_i e_j x \neq 0$, $\varphi_i(e_jx) = \varphi_i(e_j^2 e_i x)$ and $\varphi_j(e_i x) = \varphi_j(e_j^2 e_i x)$. 
Why Stembridge crystals?

**Theorem (Stembridge 2003)**

\[ B, C \text{ Stembridge crystals} \implies B \otimes C \text{ Stembridge crystal} \]
Why Stembridge crystals?

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\[ B, C \text{ Stembridge crystals} \implies B \otimes C \text{ Stembridge crystal} \]

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Every connected component of a Stembridge crystal has a unique highest weight element.
Why Stembridge crystals?

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<td>$B, B'$ Stembridge crystals, $u \in B$, $u' \in B'$ highest weight elements. If $\text{wt}(u) = \text{wt}(u')$, then $B$ and $B'$ are isomorphic.</td>
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Why Stembridge crystals?

Theorem (Stembridge 2003)

\[ B, C \text{ Stembridge crystals} \implies B \otimes C \text{ Stembridge crystal} \]

Theorem (Stembridge 2003)

Every connected component of a Stembridge crystal has a unique highest weight element.

Theorem (Stembridge 2003)

\[ B, B' \text{ Stembridge crystals, } u \in B, u' \in B' \text{ highest weight elements} \]
If \( \text{wt}(u) = \text{wt}(u') \), then \( B \) and \( B' \) are isomorphic.

Stembridge crystals describe the representation theory of the corresponding Lie algebra.
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**Conjecture** *(Assaf, Oguz 2018)*

_In addition to the Stembridge axioms, the relations below uniquely characterize queer crystals._

\[
\begin{array}{c}
\text{1} \\
\text{1} \\
\text{1} \\
\text{-1} \\
\text{1} \\
\text{-1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\end{array}
\quad
\begin{array}{c}
\text{2} \\
\text{2} \\
\text{-1} \\
\text{2} \\
\text{-1} \\
\text{-1} \\
\text{-1} \\
\text{-1} \\
\text{2} \\
\text{2} \\
\end{array}
\]
Counterexample

[Gillespie, Hawkes, Poh, S. 2018]
Question: How are the type A components glued together?
Graph on type A components

**Question**: How are the type A components glued together?

**Definition**

$\mathcal{C}$ crystal with index set $I_0 \cup \{-1\}$, $A_n$ Stembridge crystal when restricted to $I_0$

Type A graph $G(\mathcal{C})$ defined as follows:
Graph on type $A$ components

**Question:** How are the type $A$ components glued together?

**Definition**

$C$ crystal with index set $I_0 \cup \{-1\}$, $A_n$ Stembridge crystal when restricted to $I_0$

Type $A$ graph $G(C)$ defined as follows:

- **Vertices** of $G(C)$ are the type $A$ components of $C$, labeled by highest weight elements
Graph on type $A$ components

**Question:** How are the type $A$ components glued together?

**Definition**

$\mathcal{C}$ crystal with index set $I_0 \cup \{-1\}$, $A_n$ Stembridge crystal when restricted to $I_0$

Type $A$ graph $G(\mathcal{C})$ defined as follows:

- **Vertices** of $G(\mathcal{C})$ are the type $A$ components of $\mathcal{C}$, labeled by highest weight elements
- **Edge** from vertex $C_1$ to vertex $C_2$, if $\exists \ b_1 \in C_1$ and $b_2 \in C_2$ such that

$$f_{-1}b_1 = b_2.$$
Graph on type A components: example

correct graph $G(C)$
Graph on type $A$ components: example

correct graph $G(C)$

```
1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1
```

```
2 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1
```

```
1 \otimes 3 \otimes 1 \otimes 1 \otimes 2 \otimes 1
```

```
2 \otimes 3 \otimes 1 \otimes 1 \otimes 2 \otimes 1
```

```
3 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 1
```

counterexample

```
1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1
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3 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 1
```
Combinatorial description of $G(C)$

Goal

Give a combinatorial description of $G(C)$. 

Proposition (GHPS 2018)

$C_1, C_2$ distinct type $A$ components in $C$

Let $u_2 \in C_2$ be $I_0$-highest weight element

There is an edge from $C_1$ to $C_2$ in $G(C)$ if and only if $e^{-i}u_2 \in C_1$ for some $i \in I_0$. 

Combinatorial description of $G(C)$

**Goal**

Give a combinatorial description of $G(C)$.

- **Edges** described by $e_{-i}$:

**Proposition (GHPS 2018)**

$C_1, C_2$ distinct type $A$ components in $C$

Let $u_2 \in C_2$ be $l_0$-highest weight element
Combinatorial description of $G(C)$

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Give a combinatorial description of $G(C)$.

- Edges described by $e_{-i}$:

Proposition (GHPS 2018)

$C_1, C_2$ distinct type A components in $C$

Let $u_2 \in C_2$ be $l_0$-highest weight element

There is an edge from $C_1$ to $C_2$ in $G(C)$

$\iff e_{-i}u_2 \in C_1$ for some $i \in l_0$
Combinatorial description of $G(C)$ (continued)

- **Remove by-pass arrows:**
Combinatorial description of $G(C)$ (continued)

- **Combinatorial description** of remaining arrows:
  Define $f_{(-i,h)} := f_{-i}f_{i+1}f_{i+2} \cdots f_{h-1}$. 
Combinatorial description of remaining arrows:

Define \( f_{(-i,h)} := f_{-i} f_{i+1} f_{i+2} \cdots f_{h-1} \).

**Theorem (GHPS 2018)**

Let \( \mathcal{C} \) be a connected component in \( B^\otimes \ell \). Then each non by-pass edge in \( G(\mathcal{C}) \) can be obtained by \( f_{(-i,h)} \) for some \( i \) and \( h > i \) minimal such that \( f_{(-i,h)} \) applies.
Combinatorial description of $G(C)$ (continued)
Combinatorial description of $f_{-i}$

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i - 1, \ldots, 1$ from left to right
Combinatorial description of $f_{-i}$

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i-1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
Combinatorial description of $f_{-i}$

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i - 1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
- Recursively $r_j < r_{j-1}$ for $1 < j \leq i$ maximal such that $b_{r_j} = j$. 

Example $b = 1331242312111$ and $i = 3$

We overline $b_{q_j}b_q = 13312423$ and underline $b_{r_j}b_r = 1331242312111$ here $k = 1$. 
Combinatorial description of $f_{-i}$

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- Set $r_1 = q_1$
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- By definition $q_j \leq r_j$. Let $1 \leq k \leq i$ be maximal such that $q_k = r_k$.  
Combinatorial description of $f_{-i}$

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Example

$b = 1331242312111$ and $i = 3$

We overline $b_{q_j}$

$$b = \overline{1331242312111}$$
Combinatorial description of $f_-$

- $b_{qi}, b_{qi-1}, \ldots, b_{q1}$ leftmost sequence $i, i-1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
- Recursively $r_j < r_{j-1}$ for $1 < j \leq i$ maximal such that $b_{r_j} = j$.
- By definition $q_j \leq r_j$. Let $1 \leq k \leq i$ be maximal such that $q_k = r_k$.

**Example**

$b = 1331242312111$ and $i = 3$

We overline $b_{q_j}$

\[ b = 1\underline{331242312111} \]

and underline $b_{r_j}$

\[ b = 1\overline{331242312111} \]
Combinatorial description of $f_{-i}$

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i-1, \ldots, 1$ from left to right
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**Example**

$b = 1331242312111$ and $i = 3$

We overline $b_{q_j}$

$$b = 13\text{overline}3\text{overline}12423\text{overline}12111$$

and underline $b_{r_j}$

$$b = 133124\text{underline}23\text{underline}12111$$

Here $k = 1$. 
Combinatorial description of $f_{-i}$ (continued)

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i - 1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
- Recursively $r_j < r_{j-1}$ for $1 < j \leq i$ maximal such that $b_{r_j} = j$.
- By definition $q_j \leq r_j$. Let $1 \leq k \leq i$ be maximal such that $q_k = r_k$.

**Proposition**

Let $b \in B^\otimes \ell$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$. Then $f_{-i}(b)$ is obtained from $b$ by changing

- $b_{q_j} = j$ to $j - 1$ for $j = i, i - 1, \ldots, k + 1$
- $b_{r_j} = j$ to $j + 1$ for $j = i, i - 1, \ldots, k$. 
Combinatorial description of $f_{-i}$ (continued)

- $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ leftmost sequence $i, i-1, \ldots, 1$ from left to right
- Set $r_1 = q_1$
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- $b_{q_j} = j$ to $j - 1$ for $j = i, i-1, \ldots, k+1$
- $b_{r_j} = j$ to $j + 1$ for $j = i, i-1, \ldots, k$.

**Example**

$$b = 1\overline{3}31\overline{2}423\overline{1}2111$$

$$f_{-3}(b) = 1\overline{2}411\overline{4}3322111$$
Crystals of type $A_n$  

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<td>$(-1, 2)$</td>
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<td>$(-1, 2)$</td>
</tr>
<tr>
<td>$2 \otimes 1 \otimes 4 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
<td>$(-1, 2)$</td>
<td>$(-3, 4)$</td>
</tr>
<tr>
<td>$3 \otimes 1 \otimes 4 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
<td>$(-2, 3)$</td>
<td>$3 \otimes 2 \otimes 3 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
</tr>
<tr>
<td>$3 \otimes 2 \otimes 4 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
<td>$(-1, 2)$</td>
<td>$4 \otimes 2 \otimes 3 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
</tr>
<tr>
<td>$4 \otimes 3 \otimes 4 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
<td>$(-2, 4)$</td>
<td>$4 \otimes 3 \otimes 4 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1$</td>
</tr>
</tbody>
</table>

Thank you!